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FORMULATION OF THE VARIATIONAL BOUNDARY ELEMENT METHOD WITH IMPROVED ACCURACY†

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A formulation which uses Lagrange multipliers to "smooth out" discontinuities in the stress field approximation at the joints of the boundary elements, and which is in fact a hybrid formulation of the finite-element method [1, 2], is considered. The improvement to the a posteriori estimate approximation is analysed.

It is well known [1, 2] that inter-element inconsistency in the displacement field and violation of the equilibrium condition for nodal forces statically equivalent to stresses at the boundary elements, are both sources of errors in finite-element approximations. This is also true for boundary-element approximations (BEAs) of the variational boundary element method (VBEM) [3–7]. In a consistent BEA for the displacements and associated stress approximation (the latter being a derivative of the displacement approximation) there may be discontinuities in the stress field at the joins between the boundary elements.

For example, in the linear displacement approximations at points of a curvilinear boundary the stress field, being constant in the limit at each element, also has a jump at the boundaries of the elements (proportional to the value of the direction cosines of the normals at the points of the linear elements) when the VBEM algorithm is implemented [6]; we are in effect describing the approximation of a continuous function (the given stress field) by a piecewise-constant function (the required stress field). It is obvious that the desired accuracy of such an approximation may be achieved by using a sufficiently fine decomposition for the discrete boundary, which leads to a higher-order system of solving equations.

This is the basis of the duality algorithm presented below for improving the accuracy of a constrained BEA for a stress field, which, as a result of the implementation, satisfies an integral condition (as a constraint equation using Lagrange multipliers) for the equilibrium of normal stresses at the joints of elements and which results in an improvement in the accuracy of the approximation. This is established below (Sections 3 and 4).

1. Let $S_\Delta = U\Delta s_n$ $(n = 1, \ldots, N)$ be a discrete boundary (with Δs_n a boundary element (BE)) approximating the boundary S of a domain $G \subset E_m$ $(m = 2, 3)$ which may have points at infinity; it is assumed that S_Δ is a "Lipschitz continuous" boundary [1]. Following the VBEM [6] we consider an approximating variational problem for the boundary functional (BF)

$$\min_{\mathbf{u}\in D_\Delta} F_\Delta(\mathbf{u}), \quad F_\Delta = \int_{S_\Delta} \mathbf{u}\mathbf{t}^{(\nu_\Delta)}(\mathbf{u})ds_\Delta - 2\int_{S_\Delta} \mathbf{u}\mathbf{g}^{(\nu_\Delta)}ds_\Delta \tag{1.1}$$

where $\mathbf{g}^{(\nu_\Delta)}$, $\mathbf{t}^{(\nu_\Delta)}$ are the specified and required stress vectors at the points of S_Δ along the direction of the external normal $\boldsymbol{\nu}_\Delta$; the set D_Δ of kinematically admissible vector displacement functions $\mathbf{u}(x)$, $x \in G_\Delta$ is approximated [6] by discrete boundary potentials satisfying the Lamé equation (and a regularity condition at infinity if the domain G_Δ has a point at infinity).

Problem (1.1) is solvable because it is equivalent to the second problem of the theory of elasticity in the domain G_Δ with boundary S_Δ which is solvable [8] up to an arbitrary rigid displacement.

With the well-known uniqueness conditions for the solution [8] of

$$\int_{G_\Delta} \mathbf{u}dG_\Delta = 0, \quad \int_{G_\Delta} \mathbf{rot}\,\mathbf{u}dG_\Delta = 0 \tag{1.2}$$

which exclude such a displacement, it follows [6] that

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$$\min_{\mathbf{u}\in D_\Delta} F_\Delta(\mathbf{u}) = F_\Delta(\mathbf{u}_0) = d_0 = -\int_{S_\Delta} \mathbf{u}_0\mathbf{t}^{(\nu_\Delta)}(\mathbf{u}_0)dS_\Delta \tag{1.3}$$

and the minimizing element $\mathbf{u}_0$ satisfies the variational equation

$$\int_{S_\Delta} \mathbf{v}\mathbf{t}^{(\nu_\Delta)}(\mathbf{u}_0)dS_\Delta - \int_{S_\Delta} \mathbf{v}\mathbf{g}^{(\nu_\Delta)}dS_\Delta = 0 \quad \forall \mathbf{v}\in D_\Delta \tag{1.4}$$

Note that the constructions given below are assumed to be mathematically well-posed in the sense that the trace (in terms of the space $W_2^{\pm 1/2}(S_\Delta)$ theorem conditions are satisfied for functions in the Sobolev class $W_2^1(G_\Delta)$ which contains the solutions of variational problems equivalent to second-order elliptic boundary-value problems.

2. Let $s_{nn'}$ be the common boundary of adjoining BEs Δs_n and let $\Delta s_{n'}$ $(n, n' = 1, \ldots, N)$ and $\mathbf{t}_n, \mathbf{t}_{n'}$ be the values of the vector $\mathbf{t}^{(\nu_\Delta)}$ (the index ν_Δ being omitted from now on) at points $s_{nn'}$ matched to the BEA stress field at the points Δs_n, $\Delta s_{n'}$.

We define a set Λ_Δ of Lagrange multipliers $\boldsymbol{\lambda}$ in the form of sufficiently smooth vector functions, defined at points of the S_Δ, and such that the scalar function

$$f(\mathbf{u},\boldsymbol{\lambda}) = \sum_{n=1}^{N} \int_{s_{nn'}} \boldsymbol{\lambda}[\mathbf{t}_n(\mathbf{u}) - \mathbf{t}_{n'}(\mathbf{u})]ds_{nn'}$$

has the following property

$$\sup_{\boldsymbol{\lambda}\in\Lambda_\Delta} f = \begin{cases} 0, & \mathbf{t}_n(\mathbf{u}) = \mathbf{t}_{n'}(\mathbf{u}) \quad \forall n \\ +\infty, & \mathbf{t}_n(\mathbf{u}) \neq \mathbf{t}_{n'}(\mathbf{u}) \quad \forall n \end{cases} \tag{2.1}$$

This property is established by direct verification (see for example [9]), and the function f is used to construct the Lagrangian

$$L_\Delta(\mathbf{u}, \boldsymbol{\lambda}) = F_\Delta(\mathbf{u}) - 2f(\mathbf{u}, \boldsymbol{\lambda}) \tag{2.2}$$

The interpretation of the Lagrange multipliers follows from the physical significance of the terms occurring in the functional L_Δ: each term represents the work performed by the unknown and specified surface stresses during the displacements, and, in particular, the second term is the work done by the stress jumps at the joins of adjacent BEs.

The equivalence of the problem

$$\min_{\mathbf{u}\in D_\Delta} \max_{\boldsymbol{\lambda}\in\Lambda_\Delta} L_\Delta(\mathbf{u},\boldsymbol{\lambda})$$

(with the condition that the minimum with respect to $\mathbf{u}$ and the maximum with respect to $\boldsymbol{\lambda}$ are reached) to the original problem (1.1) is established as in [7]: if $\mathbf{t}_n(\mathbf{u}) = \mathbf{t}_{n'}(\mathbf{u})$, which indicates the continuity of the stress fields at the points of S_Δ, then $\max_{\boldsymbol{\lambda}} f = 0$ (see (2.1)) and $\min_{\mathbf{u}}\max_{\boldsymbol{\lambda}} L_\Delta = \min_{\mathbf{u}} F_\Delta(\mathbf{u}) = d_0$ (see (1.3)).

The dual problem

$$\max_{\boldsymbol{\lambda}\in\Lambda_\Delta} \min_{\mathbf{u}\in D_\Delta} L_\Delta(\mathbf{u},\boldsymbol{\lambda})$$

has meaning if

$$\min_{\mathbf{u}\in D_\Delta} \max_{\boldsymbol{\lambda}\in\Lambda_\Delta} L_\Delta = \max_{\boldsymbol{\lambda}\in\Lambda_\Delta} \min_{\mathbf{u}\in D_\Delta} L_\Delta = d_0 \tag{2.3}$$

We will prove the right-hand equality in (2.3). Let $\boldsymbol{\lambda}$ be fixed; then the solution $\mathbf{u}_{\boldsymbol{\lambda}} = \mathbf{u}(\boldsymbol{\lambda})$ of the problem $\min_{\mathbf{u}} L_\Delta(\mathbf{u}, \boldsymbol{\lambda})$ is obtained from the equation ($\forall\, \mathbf{v} \in D_\Delta$)

$$\mathbf{grad}_{\mathbf{u}} L_\Delta = \mathbf{grad}_{\mathbf{u}} F_\Delta(\mathbf{u}_{\boldsymbol{\lambda}}) - 2f(\mathbf{v},\boldsymbol{\lambda}) = 0 \tag{2.4}$$

When $\mathbf{v} = \mathbf{u}_\lambda$, we obtain, using (2.2), the value of the functional L_Δ at the solution $\mathbf{u}_\lambda$ (with the multiplier λ fixed)

$$\min_{\mathbf{u}} L_\Delta = L_\Delta(\mathbf{u}_\lambda, \boldsymbol{\lambda}) = -f(\mathbf{u}_\lambda, \boldsymbol{\lambda}) - \int_{S_\Delta} \mathbf{u}_\lambda \mathbf{g}^{(\mathbf{v}_\Delta)} dS_\Delta \tag{2.5}$$

The dual problem then reduces to the problem

$$\max_{\boldsymbol{\lambda}} L_\Delta(\mathbf{u}_\lambda, \boldsymbol{\lambda}) = -\min_{\boldsymbol{\lambda}}[-L_\Delta(\mathbf{u}_\lambda, \boldsymbol{\lambda})] \tag{2.6}$$

The second condition is then used to determine the saddle point of the Lagrangian L_Δ (the first condition being condition (2.4))

$$\mathbf{grad}_\lambda L_\Delta(\mathbf{u}, \boldsymbol{\lambda}) = 2f(\mathbf{u}, \boldsymbol{\mu}) = 0 \quad \forall \boldsymbol{\mu} \in \Lambda_\Delta \tag{2.7}$$

When $\boldsymbol{\mu} = \boldsymbol{\lambda}$ and $\mathbf{u} = \mathbf{u}_\lambda$ it follows from this that $f(\mathbf{u}_\lambda, \boldsymbol{\lambda}) = 0$ and from (2.5) we obtain

$$-L_\Delta(\mathbf{u}_\lambda, \boldsymbol{\lambda}) = \int_{S_\Delta} \mathbf{u}_\lambda \mathbf{g}^{(\mathbf{v}_\Delta)} ds_\Delta \quad \forall \boldsymbol{\lambda} \in \Lambda_\Delta$$

Then, when $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0$ and $\mathbf{u}_{\lambda 0} = \mathbf{u}_0$, it follows from (2.6) and (1.4) that when $\mathbf{v} = \mathbf{u}_0$

$$\max_{\boldsymbol{\lambda}} \min_{\mathbf{u}} L_\Delta = -\int_{S_\Delta} \mathbf{u}_0 \mathbf{g}^{(\mathbf{v}_\Delta)} ds_\Delta = -\int_{S_\Delta} \mathbf{u}_0 \mathbf{t}^{(\mathbf{v}_\Delta)}(\mathbf{u}_0) ds_\Delta = d_0$$

(see (1.3)), which proves the duality relation (2.3), and this implies the existence of the saddle point ($\mathbf{u}_0$, $\boldsymbol{\lambda}_0$) of the Lagrangian (2.2) and a two-sided estimate of the value of the functional at the exact solution

$$L_\Delta(\mathbf{u}_0, \boldsymbol{\lambda}) \leq L_\Delta(\mathbf{u}_0, \boldsymbol{\lambda}_0) = F_\Delta(\mathbf{u}_0) \leq L_\Delta(\mathbf{u}, \boldsymbol{\lambda}_0) \quad \forall \mathbf{u} \in D_\Delta, \quad \boldsymbol{\lambda} \in \Lambda_\Delta \tag{2.8}$$

This condition is satisfied by the solution of the system of variational equations (2.4) and (2.7).

Before proceeding to the approximation and the discrete problem we note that the above establishment of the equivalence of the original problem and the problem transformed by the Lagrange multiplier method leads to the same solution in both the exact case and the approximate case with any level of accuracy. Below we show that the duality algorithm based on the equivalence theorem enables us to refine the approximate solution.

3. We will now proceed to the discrete problem for Eqs (2.4) and (2.7), which is essentially the solution of the dual problem of searching for the saddle point $L_\Delta(\mathbf{u}, \boldsymbol{\lambda})$. We consider the case of the numerically easily implemented linear (bilinear when $m = 3$) isoperimetric BEA for the boundary S_Δ and approximate solution ($\mathbf{u}_N$, $\boldsymbol{\lambda}_N$). Let

$$y_\Delta = \Sigma_n \Sigma_k \mathbf{Y}_{nk} \psi_k, \qquad k = 1,\dots, K; \quad n = 1,\dots, N$$

be the parametric equation for S_Δ, where $\mathbf{Y}_{nk}$ is the m-dimensional vector of global coordinates for the nodes of the decomposition of S_Δ

$$\mathbf{u}_N = \Sigma_n \Sigma_k \mathbf{U}_{nk} \psi_k, \quad \boldsymbol{\lambda}_N = \Sigma_n \Sigma_k \boldsymbol{\Lambda}_{nk} \psi_k \tag{3.1}$$

are the global interpolation functions for the displacement field continuous at the points of S_Δ, where $\mathbf{U}_{nk}$, $\boldsymbol{\Lambda}_{nk}$ are m-dimensional vectors of the nodal values of the approximate solution, $\psi_k(\eta)$ are the linear (bilinear when $m = 3$) basis functions of the BEM [10] and $y_n(\eta)$ connect the global and local coordinates at the points of $\Delta s_{n=1,\dots,N}$.

The discrete analogues of Eqs (2.4) and (2.7) at the approximations (3.1) correspond to the equations

$$\mathbf{grad}_{\mathbf{U}_{nk}} L_\Delta(\mathbf{u}_N, \boldsymbol{\lambda}_N) = 0, \quad \mathbf{grad}_{\boldsymbol{\Lambda}_{nk}} L_\Delta(\mathbf{u}_N, \boldsymbol{\lambda}_N) = 0$$

and when the matrices are constructed the systems of discrete equations over the elements of Δs_n are written in the form

$$\sum_{n,n'\in\{N\}}\left\{\sum_{\{n'\}}\mathbf{U}_{n'l}c_{kl}^{n'}-\sum_{\{n'\}}\mathbf{Q}_{n'l}g_{kl}^{n'}-\mathbf{\Lambda}_{nk}\sum_{\{n'\}}(\bar{c}_{kl}^{n}-\bar{c}_{kl}^{n'})\right\}=0 \quad (3.2)$$

$$\sum_{n,n'\in\{N\}}\left\{\sum_{\{n'\}}(\mathbf{U}_{nl}\bar{c}_{kl}^{n}-\mathbf{U}_{n'l}\bar{c}_{kl}^{n'})\right\}=0 \quad (3.3)$$

Here $\{n'\}$ is the set of adjacent BEs sharing the node $k\in\Delta s_n$; the contributions of the adjacent BEs depending on the required and specified stress field with an interpolant represented in the form $\mathbf{g}_n^{(\mathbf{v}_n)}=\Sigma\mathbf{Q}_{nk}\,\psi_k$ $(k=1,\ldots,K)$ are given by

$$c_{kl}^{n'}=\int_{\Delta s_{n'}}\psi_k T_{n'}\psi_l|J_n|ds_{n'}(\eta),\quad g_{kl}^{n'}=\int_{\Delta s_{n'}}\psi_k\psi_l|J_{n'}|ds_{n'}(\eta)$$

where T_n is the scalar operator corresponding [6] to the approximation $\mathbf{t}^{(\mathbf{v}_n)}(\mathbf{u}_n)$, and $|J_n|$ is the Jacobian of the transformation $y_n(\eta)$; $\bar{c}_{kl}^{n'}$ are the values of the contributions of $c_{kl}^{n'}$ for fixed coordinates η at the boundaries Δs_n. For a bilinear BEA: $s_{nn'}$ is a line, the integral over $s_{nn'}$ corresponds to an integral over η_i, $i=1,2$ (with fixed η_j, $j=1,2$), and $\bar{T}_{n'}$ is the trace of the operator $T_{n'}$ on $s_{nn'}$. For a linear BEA: $s_{nn'}$ is the node point k (the common point of adjacent BEs) and $\bar{c}_{kl}^{n'}=\psi_k\bar{T}_{n'}\psi_l$ where $\psi_k=1$ and $\eta=\pm1$.

For a linear BEA for nodes $k,l=1,2$ of the nth BE the terms in braces in (3.2) are written in the form $(\{n'\}=\{n-1,n+1\})$

$$\Big\{\Big[\mathbf{U}_{(n-1)2}c_{21}^{n-1}+\mathbf{U}_{(n+1)1}c_{12}^{n+1}\Big]-\Big[\mathbf{Q}_{(n-1)2}g_{21}^{n-1}+\mathbf{Q}_{(n+1)1}g_{12}^{n+1}\Big]-$$
$$-\Big[\mathbf{\Lambda}_{n1}(\bar{c}_{12}^{n}-\bar{c}_{21}^{n-1})+\mathbf{\Lambda}_{n2}(\bar{c}_{21}^{n}-\bar{c}_{12}^{n+1})\Big]\Big\}$$

The sum $\Sigma_{n'}$ in (3.3) is correspondingly written in the form

$$\{(\mathbf{U}_{n1}\bar{c}_{12}^{n}-\mathbf{U}_{(n-1)2}\bar{c}_{21}^{n-1})+(\mathbf{U}_{n2}\bar{c}_{21}^{n}-\mathbf{U}_{(n+1)1}\bar{c}_{12}^{n+1})\}$$

The matching conditions $\mathbf{U}_{n1}=\mathbf{U}_{(n-1)2}$, $\mathbf{U}_{n2}=\mathbf{U}_{(n+1)1}$ for the nodal values of the displacements are then used, followed by summation over n in (3.2) and (3.3). The formation of the system of boundary equations for the bilinear BEA is more complicated.

The algorithm for implementing the solution of system (3.2), (3.3) is as follows: the nodal values $\mathbf{U}_{n'l}$ are determined from (3.2) in terms of $\mathbf{Q}_{n'l}$, $\mathbf{\Lambda}_{nk}$ ($\mathbf{Q}_{n'l}$ are the specified nodal values) and then the values of $\mathbf{\Lambda}_{nk}$ are determined from (3.3) with the condition $\mathbf{U}_{nl}=\mathbf{U}_{n'l}\,\forall n\in\{N\}$ (the continuity condition for the displacement field at the points of S_Δ).

We shall clarify in what sense the approximation to the solution obtained by implementing the above duality algorithm is an improvement. For every fixed λ the solution amounts to a sum $\mathbf{u}_{Ng}+\mathbf{u}_{N\lambda}$, which follows from Eq. (2.4) or from its discrete analogue (3.2). Here $\mathbf{u}_{Ng}$ is the "Ritz" solution of problem (1.1) and $\mathbf{u}_{N\lambda}$ is the solution of the functional problem (see (2.2))

$$L_\Delta^*(\mathbf{u})=\int_{S_\Delta}\mathbf{ut}^{(\mathbf{v}_\Delta)}(\mathbf{u})ds_\Delta-2f(\mathbf{u},\boldsymbol{\lambda})$$

on the set D_Δ, where $\boldsymbol{\lambda}$ can be considered as a displacement field specified at the points S_Δ. It is clear that $\mathbf{u}_{N\lambda}\to0$ when $N\to\infty$ because when the decomposition of S_Δ is refined the stress jumps on Δs_n at the joins of the elements are "smoothed", so that the jump function $f(\mathbf{u}_N,\boldsymbol{\lambda})\to0$ (see below) and in the limit we obtain $\mathbf{u}_{N\lambda}=0$.

This follows from the condition $\mathbf{grad}_\mathbf{u}L_\Delta^*(\mathbf{u}_{N\lambda})=0$ (when $f(\mathbf{u}_N,\boldsymbol{\lambda})\to0$), which is written in the form $\int_{S_\Delta}\mathbf{vt}^{(\mathbf{v}_\Delta)}(\mathbf{u}_{N\lambda})ds_\Delta=0\ \forall\,\mathbf{v}\in D_\Delta$.

From this we obtain $\mathbf{t}^{(\mathbf{v}_\Delta)}(\mathbf{u}_{N\lambda})=0$ and consequently $\mathbf{u}_{N\lambda}|_{S_\Delta}=c'$, where c is a constant. Let $\bar{\mathbf{u}}_{N\lambda}(x)=c$ be the solution when $x\in\bar{G}_\Delta$. (Here $\bar{\mathbf{u}}_{N\lambda}\in D_\Delta$ (see (1.1)) because the Lamé equation is satisfied.) Then the constant satisfying (1.2) is identically equal to zero. (Conditions (1.2) should be satisfied by the sum $\mathbf{u}_{Ng}+\mathbf{u}_{N\lambda}$, which also means that it is satisfied by each term separately.)

Accordingly, when $N\to\infty$ we have $\mathbf{u}_{Ng}\to\mathbf{u}_0$ [5, 6] where $\mathbf{u}_0$ is the exact solution (see (1.3)); finally, when $N\to\infty$ we obtain $(\mathbf{u}_{Ng}+\mathbf{u}_{N\lambda})\to\mathbf{u}_0$. Thus $\mathbf{u}_{N\lambda}$ can be considered to be an improvement to the solution $\mathbf{u}_{Ng}$ associated with the "smoothing" of the stress field discontinuities at the joints of the BEs. The fact that the stated correction improves the approximation is confirmed below by the better a posteriori error estimate compared with the estimate obtained [6] in the formulation of the VBEM which takes no account of the stress jump at the BE joins. We recall that the estimate is improved if for a fixed decomposition (i.e. for fixed N) the right-hand side of the estimate decreases.

4. The norm in the space $W_2^{1/2}(S_\Delta)$ (or $L_2(S\Delta)$) of the difference $\mathbf{u}_0 - \mathbf{u}_N$ is estimated [6] in terms of the difference in the values of the functionals in the direct and dual problems in the approximate solutions

$$0 < \delta_{1N} \equiv \frac{1}{2}[F_\Delta(\mathbf{u}_N) - \Phi_\Delta(\mathbf{t}^{(\nu_\Delta)}(\mathbf{u}_N))] = \int_{S_\Delta} \mathbf{u}_N \mathbf{t}^{(\nu_\Delta)}(\mathbf{u}_N) ds_\Delta - \int_{S_\Delta} \mathbf{u}_N \mathbf{g}^{(\nu_\Delta)} ds_\Delta \tag{4.1}$$

A similarly stated norm of the difference can be estimated [7] on the basis of estimate (2.8) in terms of the difference of functionals

$$\begin{aligned} 0 < \delta_{2N} &\equiv [L_\Delta(\mathbf{u}_N, \boldsymbol{\lambda}_0) - L_\Delta(\mathbf{u}_0, \boldsymbol{\lambda}_N)] = \\ &= -\int_{S_\Delta} \mathbf{u}_N \mathbf{t}^{(\nu_\Delta)}(\mathbf{u}_N) ds_\Delta + \int_{S_\Delta} \mathbf{u}_N \mathbf{g}^{(\nu_\Delta)} ds_\Delta + f(\mathbf{u}_N, \boldsymbol{\lambda}_N) \end{aligned} \tag{4.2}$$

Here the value $L_\Delta(\mathbf{u}_N, \boldsymbol{\lambda}_0)$ and $L_\Delta(\mathbf{u}_0, \boldsymbol{\lambda}_N)$ are determined from (2.4) and (2.2) when $\mathbf{u} = \mathbf{u}_N$, $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0$ and when $\mathbf{u} = \mathbf{u}_0$, $\lambda = \boldsymbol{\lambda}_N$.

Respectively adding and subtracting the left- and right-hand sides of (4.2) and (4.3), we obtain

$$\delta_{2N} + \delta_{1N} = f(\mathbf{u}_N, \boldsymbol{\lambda}_N) > 0 \tag{4.3}$$

$$\delta_{2N} - \delta_{1N} = f(\mathbf{u}_N, \boldsymbol{\lambda}_N) - 2\left[\int_{S_\Delta} \mathbf{u}_N \mathbf{t}^{(\nu_\Delta)}(\mathbf{u}_N) ds_\Delta - \int_{S_\Delta} \mathbf{u}_N \mathbf{g}^{(\nu_\Delta)} ds_\Delta\right] \tag{4.4}$$

(where (4.3) has the physical meaning of the work done by the stress discontinuities over the approximating displacements).

The second term in (4.4) is $\mathbf{grad}_\mathbf{u} F_\Delta(\mathbf{u}_N)$, and from (2.4) when $\mathbf{u}_\lambda = \mathbf{u}_N$ and $\mathbf{v} = \mathbf{u}_N$ we have

$$\mathbf{grad}_\mathbf{u} F_\Delta(\mathbf{u}_N) = 2f(\mathbf{u}_N, \boldsymbol{\lambda}_N)$$

Then from (4.4) we obtain by virtue of (4.3)

$$\delta_{2N} - \delta_{1N} = -f(\mathbf{u}_N, \boldsymbol{\lambda}_N) < 0$$

This leads to an improvement in the a posteriori estimate for the approximations $\{\mathbf{u}_N\}$

$$[L_\Delta(\mathbf{u}_N, \boldsymbol{\lambda}_0) - L_\Delta(\mathbf{u}_0, \boldsymbol{\lambda}_N)] < \tfrac{1}{2}[F_\Delta(\mathbf{u}_N) - \Phi_\Delta(\mathbf{t}^{(\nu_\Delta)}(\mathbf{u}_N))]$$

Because of the proven convergence [5] $\mathbf{u}_N \to \mathbf{u}_0$ when $N \to \infty$, it follows simultaneously from (4.1) and (4.2) that the estimates under consideration have an asymptotic nature, i.e. δ_{1N}, $\delta_{2N} \to 0$ when $N \to \infty$: the right-hand side of Eq. (4.1) tends to zero by virtue of (1.4) when $N \to \infty$; the right-hand side of Eq. (4.2) tends to zero by virtue of (1.4) and because $f(\mathbf{u}_N, \boldsymbol{\lambda}_N) \to 0$ when $N \to \infty$ (see above). The convergence $f(\mathbf{u}_N, \boldsymbol{\lambda}_N) \to 0$ when $N \to \infty$ can also be established analytically: using the technique of trace theorem estimates and the methods of [7, 11] it can be shown that $f(\mathbf{u}_N, \boldsymbol{\lambda}_N) \to f(\mathbf{u}_0, \boldsymbol{\lambda}_0)$ when $N \to \infty$.

Thus the above algorithm enables us to reduce the effect on the accuracy of the BEM approximations of inconsistencies in the stress field generated by the coupled approximation of the stress and displacement fields.

An alternative formulation of the VBEM [7] using unconstrained approximations of the displacement and stress fields enables one to implement a consistent approximation of the stress field (continuous at the nodal points). According to the accuracy estimate of these approximations the estimates considered above are identical [7]. We also note that the a posteriori estimates discussed here are obtained on the basis of duality relations, similar to the estimates which are obtained using opposed variational methods (relative to energy methods) of the Trefftz type [8, 12].

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